

Admissibility and Exact Observability of Observation Operators for Micro-Beam Model: Time and Frequency Domain Approaches

Mohammad S. Edalat zadeh, Aria Alasty, and Ramin Vatankhah, *Member, IEEE*

Abstract—This study focuses on the exact observability of a non-classical Euler-Bernoulli micro-beam equation. This non-classical model was derived based on the strain gradient elasticity theory, which is intended to explain the phenomenon of size effect at the micron scale. Spectral properties of the corresponding state operator are studied; an asymptotic expression for eigenvalues is calculated, and eigenfunctions are analyzed in order to check the necessary conditions for the exact observability of the system. By examining the eigenfunctions, it is shown that among non-collocated boundary outputs, only measurement of the non-classical moment at the root of the beam yields an admissible observation operator and also defines an exactly observable system. An alternative proof based on the multiplier method, which is commonly employed in the literature on the observability and controllability of infinite dimensional dynamical systems, is presented to provide a comparison between the time and frequency domain approaches.

Index Terms—MEMS, flexible structures, distributed parameter systems, observability.

I. INTRODUCTION

ONE of the most important structural components in an atomic force microscope [1] and many micro-electromechanical systems (MEMS) [2] is the micro cantilever beam. High efficiency and simple manufacturing process of micro-cantilever beams make them to play a significant role in MEMS devices.

In the last two decades, many experimental observations in some metals and polymers have demonstrated that the classical continuum mechanics cannot yield accurate static and dynamic models for micro-scale structures [3]. In this way, investigators proposed non-classical continuum theories to accurately predict the static and dynamic behaviors of micro-scale structures. Modified strain gradient theory or briefly strain gradient theory is one of the most successful and inclusive non-classical continuum theories introduced by Lam et al. in 2003 [4]. Recently, this newly established theory has been extensively utilized to study static and dynamic behaviors of the micro-scale beams. Below, some of these works are outlined briefly.

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M.S. Edalat zadeh and A. Alasty (Corresponding Author) are with Department of Mechanical Engineering, Sharif University of Technology, Azadi Ave., Tehran, Iran (e-mails: sajad_edalat@mech.sharif.edu and aalasty@sharif.edu).

R. Vatankhah, is with the Department of Mechanical Engineering, Shiraz University, Shiraz, Iran (e-mail: rvatankhah@shirazu.ac.ir).

In 2009, Kong et al. [5] derived a new governing partial differential equation of motion of vibrating Euler-Bernoulli micro-scale beams by using the strain gradient elasticity theory introduced by Lam et al. In a similar way utilized by Wang et al. [6] in 2010, a governing equation of motion of strain gradient Timoshenko micro-beams was formulated. Considering the effect of mid-plane stretching, Kahrobaiyan et al. developed a nonlinear Euler-Bernoulli beam model based on strain gradient elasticity theory in 2011 [7]. In 2013, forced vibrations of geometrically nonlinear strain gradient Euler-Bernoulli beams were investigated by Vatankhah et al. [8] utilizing perturbation techniques. In 2014, by truncating the governing partial differential equation, Edalat zadeh et al. [9, 10] studied stabilization of the non-classical strain gradient Euler-Bernoulli beams, subjected to some nonlinear distributed forces affecting micron and sub-micron structures. Omitting nonlinear terms, but without resorting to model truncation, the authors [11, 12] investigated the boundary stabilization and exact controllability of the previous beam model. In these studies, well-posedness, stabilizability, and exact controllability of the closed-loop system have been proven by using operator theory and semigroup techniques. In a recent study, Guzmán and Zhu [13] proved the exact controllability of the same micro-beam model considering only one control input.

The concept of observability for infinite dimensional dynamical systems has received considerable attention in recent years. The survey paper of Lagnese [14] and the general exposition of Bensoussan [15] provided a comprehensive study in this field. Dolecki and Russell [16] showed that the concept of exact observability for an infinite dimensional dynamical system is dual to that of exact controllability. Furthermore, the duality between an admissible observation operator and an admissible control operator in this framework was introduced by Salamon [17]. A general necessary condition for exact observability of the systems governed by partial differential equations (PDEs) was then obtained by Russell and Weiss [18]. To tackle an observability or controllability problem, most researchers adopt a time domain approach in which the governing PDE or its dual counterpart is manipulated in various ways to meet the necessary conditions for the observability or controllability. These ways include the following: multiplier method [19], microlocal analysis technique [20], and nonharmonic Fourier series [21]. The introduction of the Hautus-test for infinite dimensional dynamical systems established a basis for the frequency domain approach, which has rarely been adopted by researcher. Liu et al. [22] obtained

a Hautus-type criterion for the exact controllability of systems with bounded input operators; they then applied the criterion to several elastic systems. Recently, a criterion was established for unbounded observation operators with application to the Schrödinger equation [23]. In addition, the Hautus-type test has been further developed to characterize the exact observability of a system only in terms of observation operators and spectral elements of state operators [24].

Although the controllability and control design problems in flexible structure models such as string, rod, beam, and plate models have been addressed in many studies (e.g. see [13, 25–29]), the observability and observer design problems of such models have rarely been investigated in the literature. In 2005, backstepping-based infinite dimensional boundary observers were designed by Smyshlyaev and Kirtic [30] to stabilize a class of one-dimensional parabolic PDEs. In 2008, Nguyen [31] designed an observer with a finite number of measurements for second order infinite dimensional systems. Guo et al. studied the exponential stabilization of a one-dimensional wave equation by a boundary controller with collocated and non-collocated observations in [32] and [33], respectively. In another study in 2008 [34], they proposed a boundary force control and bending strain measurement to stabilize a classical Euler-Bernoulli beam model. In this paper, the boundary actuator is attached at the free end of the beam while the bending strain observation occurred at the clamped end. In 2011, infinite-dimensional Luenberger-like observers were suggested for a vibrating rotating classical Euler-Bernoulli beam system with constant angular velocity by Li and Xu [35]. In 2012, Dogan and Morgul [36] achieved the same goal as the previous article but by using a backstepping boundary observer.

This paper extends our recent studies on the boundary stabilization and controllability of the non-classical Euler-Bernoulli micro-beam under collocated controls [11, 12] to address the non-collocated observation problems. Although the controllability and observability analyses of a collocated closed-loop system are performed straightforwardly, the performance of such systems may not be satisfactory [37]. We approached the stabilization problem in [11] by designing a boundary control law and constructing a suitable Lyapunov function for the collocated control system; however, this method is not effective for stabilizing a non-collocated control system. Operator semigroup theory provides some methods such as Riesz basis approach and spectral analysis to deal with such systems.

The rest of this paper is organized as follows. Section 2 provides a brief description of the non-classical Euler-Bernoulli micro-beam model. Section 3 is devoted to the orthonormal basis and spectrum of the beam state operator; an asymptotic expression for eigenvalues is derived in this section for use in the next section. Section 4 discusses the exact observability of the system for various output operators; in addition, it provides a comparison between time and frequency domain approach used to prove the exact observability of the system. Finally, some concluding remarks are given in section 5.

II. MICRO-BEAM MODEL

According to Lam et al. [4], a more realistic, though more complicated, formulation for flexible micro-scale beams can be derived from modified strain gradient elasticity theory using Euler-Bernoulli beam assumptions and Hamilton's principle. As a result, for a micro-cantilever beam with uniform cross-section A and length L , the governing PDE of motion and corresponding boundary conditions (BCs) are derived as follows:

$$K_1 \frac{\partial^4 \hat{w}}{\partial \hat{x}^4} - K_2 \frac{\partial^6 \hat{w}}{\partial \hat{x}^6} + \rho A \frac{\partial^2 \hat{w}}{\partial \hat{t}^2} = 0, \quad (1)$$

$$\begin{cases} \hat{w}(0, \hat{t}) = \frac{\partial \hat{w}}{\partial \hat{x}}(0, \hat{t}) = \frac{\partial^2 \hat{w}}{\partial \hat{x}^2}(0, \hat{t}) = 0, \\ K_2 \frac{\partial^5 \hat{w}}{\partial \hat{x}^5}(L, \hat{t}) - K_1 \frac{\partial^3 \hat{w}}{\partial \hat{x}^3}(L, \hat{t}) = \hat{F}, \\ K_1 \frac{\partial^2 \hat{w}}{\partial \hat{x}^2}(L, \hat{t}) - K_2 \frac{\partial^4 \hat{w}}{\partial \hat{x}^4}(L, \hat{t}) = \hat{M}^c, \\ K_2 \frac{\partial^3 \hat{w}}{\partial \hat{x}^3}(L, \hat{t}) = \hat{M}^{nc}, \end{cases} \quad (2)$$

where \hat{x} and \hat{t} denote the spatial and time variables, respectively; ρ is the beam density; $\hat{w}(\hat{x}, \hat{t})$ indicates the lateral deflection; \hat{F} , \hat{M}^c , and \hat{M}^{nc} can be considered as control inputs and refer to boundary force, moment, and non-classical moment exerted at the tip of the beam, respectively. In addition,

$$\begin{aligned} K_1 &= EI + \mu A \left(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2 \right), \\ K_2 &= \mu I \left(2l_0^2 + \frac{4}{5}l_1^2 \right), \end{aligned} \quad (3)$$

where I is the area moment of inertia of the beam cross-section; E and μ denote Young and shear modulus, respectively; l_0 , l_1 , and l_2 are additional material constants associated with higher order stress tensors. It can be observed that setting l_0 , l_1 , and l_2 to zero leads to the classical Euler-Bernoulli beam model. For the sake of brevity, the following dimensionless variables are introduced:

$$\begin{aligned} w &= \frac{\hat{w}}{L}, \quad x = \frac{\hat{x}}{L}, \quad t = \sqrt{\frac{K_1}{\rho A L^4}} \hat{t}, \quad \zeta = \frac{K_2}{K_1 L^2}, \\ F &= \frac{L^2}{K_1} \hat{F}, \quad M^c = \frac{L}{K_1} \hat{M}^c, \quad M^{nc} = \frac{1}{K_1} \hat{M}^{nc}. \end{aligned} \quad (4)$$

From now on, all variables are dimensionless. By applying the dimensionless variables to the governing equations, the following PDE and corresponding BCs are obtained:

$$\frac{\partial^4 w}{\partial x^4} - \zeta \frac{\partial^6 w}{\partial x^6} + \frac{\partial^2 w}{\partial t^2} = 0. \quad (5)$$

$$\begin{cases} w(0, t) = \frac{\partial w}{\partial x}(0, t) = \frac{\partial^2 w}{\partial x^2}(0, t) = 0, \\ \zeta \frac{\partial^5 w}{\partial x^5}(1, t) - \frac{\partial^3 w}{\partial x^3}(1, t) = F, \\ \frac{\partial^2 w}{\partial x^2}(1, t) - \zeta \frac{\partial^4 w}{\partial x^4}(1, t) = M^c, \\ \zeta \frac{\partial^3 w}{\partial x^3}(1, t) = M^{nc}. \end{cases} \quad (6)$$

Well-posedness of the above PDE with corresponding BCs has been proved by Vatankeh et al. in 2013 [11]. Moreover, referring to Kong et al. [5], the dimensionless kinetic energy K and the strain energy U of this physical system can be determined from

$$\begin{aligned} K &= \frac{1}{2} \int_0^1 \left(\frac{\partial w}{\partial t} \right)^2 dx, \\ U &= \frac{1}{2} \int_0^1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \zeta \left(\frac{\partial^3 w}{\partial x^3} \right)^2 dx. \end{aligned} \quad (7)$$

It should be noted that the sum of kinetic and strain energy (i.e. $E = K + U$) is an invariant of the system with zero inputs.

III. ORTHONORMAL BASIS

In order to be able to use the frequency domain approach, the spectral properties of the state operator corresponding to the beam model are investigated in this section. In the first place, let's consider operator $A_0 : D(A_0) \subset H \rightarrow H$ defined as follows:

$$\begin{aligned} A_0 f &= f^{(4)} - \zeta f^{(6)}, \\ D(A_0) &= \left\{ f \in H^6(0,1) \cap H_E^3(0,1) \mid f^{(2)}(1) - \zeta f^{(4)}(1) \right. \\ &\quad \left. = \zeta f^{(5)}(1) - f^{(3)}(1) = \zeta f^{(3)}(1) = 0 \right\}, \end{aligned} \quad (8)$$

where $f^{(m)}$ stands for the m th order derivative of f with respect to x ; the Sobolev space $H^k(0,1)$ consists of all functions whose derivatives up to order $k-1$ are absolutely continuous and the k th order derivative has finite L^2 norm; In addition,

$$H := L^2(0,1) = \{f : [0,1] \rightarrow \mathbb{R} \mid \int_0^1 f^2 dx < \infty\}, \quad (9)$$

$$H_E^3(0,1) := \{f \in H^3(0,1) \mid f(0) = f^{(1)}(0) = f^{(2)}(0) = 0\}.$$

Theorem 3.1. The unbounded operator A_0 admits an infinite set of eigenvalues which are positive and increasing; furthermore, the corresponding eigenfunctions form an orthonormal basis of H .

The spectral properties of a self-adjoint operator with a compact resolvent are characterized in [38, Sec. 3.2]. Accordingly, the following lemmas should be proven at first in order to prove the theorem.

Lemma 3.1. A_0 is a symmetric and strictly positive operator.

Proof: For every $f, g \in D(A_0)$, a repeated integration by parts gives

$$\begin{aligned} \langle A_0 f, g \rangle &= \int_0^1 (f^{(4)} - \zeta f^{(6)}) g dx \\ &= \left[f^{(3)} g - f^{(2)} g^{(1)} - \zeta f^{(5)} g + \zeta f^{(4)} g^{(1)} - \zeta f^{(3)} g^{(2)} \right]_0^1 \\ &\quad + \int_0^1 (f^{(2)} g^{(2)} + \zeta f^{(3)} g^{(3)}) dx, \\ \langle f, A_0 g \rangle &= \int_0^1 (g^{(4)} - \zeta g^{(6)}) f dx \\ &= \left[g^{(3)} f - g^{(2)} f^{(1)} - \zeta g^{(5)} f + \zeta g^{(4)} f^{(1)} - \zeta g^{(3)} f^{(2)} \right]_0^1 \end{aligned}$$

$$+ \int_0^1 (g^{(2)} f^{(2)} + \zeta g^{(3)} f^{(3)}) dx.$$

According to the definition of the domain of A_0 , the boundary terms in these equations vanish, giving

$$\langle A_0 f, g \rangle = \langle f^{(2)}, g^{(2)} \rangle + \zeta \langle f^{(3)}, g^{(3)} \rangle = \langle f, A_0 g \rangle$$

which shows that operator A_0 is symmetric. It can also be seen that $\langle A_0 f, f \rangle = \|f^{(2)}\|_2^2 + \zeta \|f^{(3)}\|_2^2 \geq 0$; hence, A_0 is indeed a strictly positive operator. \square

Lemma 3.2. A_0 is surjective and A_0^{-1} is a compact operator.

Proof: From the theory of linear ordinary differential equations (ODEs), for any $h \in H$, solving $A_0 f = h$ for f leads to

$$f(x) = \int_0^1 G(x,s) h(s) ds, \quad (10)$$

where $G(x,s)$ is known as the Green's function, which is the solution of $A_0 G(x,s) = \delta(x-s)$. It is derived as:

$$G(x,s) = \begin{cases} -c_3(s) - c_2(s) - \zeta^{-1/2} (c_2(s) - c_3(s)) x \\ \quad - \zeta^{-1} (c_2(s) + c_3(s)) x^2/2 + c_1(s) x^3 \\ \quad + c_2(s) e^{x\zeta^{-1/2}} + c_3(s) e^{-x\zeta^{-1/2}}, & x \leq s, \\ 2e\zeta^{-1/2} (\zeta^{-1/2} - 1) c_4(s) + c_6(s) \\ \quad + c_5(s)x + e\zeta^{-1/2} (\zeta^{-3/2} - \zeta^{-1}) c_4(s) x^2 \\ \quad + c_4(s) e^{x\zeta^{-1/2}} + c_4(s) e^{(2-x)\zeta^{-1/2}}, & x \geq s. \end{cases}$$

In the above equation, the functions $c_i(s)$, $i = 1, 2, \dots, 6$, can be uniquely derived. Briefly, they are adjusted such that at $x = s$ the Green's function and its derivatives with respect to x up to order four are continuous; in addition, its fifth order derivative must have a jump $-1/\zeta$ at this point. As a result, the Green's function of the operator A_0 has a finite L^2 norm; hence, the operator A_0 is surjective. Furthermore, A_0^{-1} maps H into a dense subset of $H^6(0,1)$ which is compactly embedded in H by the Rellich-Kondrachov compact embedding theorem [39, Ch. 6]. Hence, A_0^{-1} is compact on H , and the proof is complete. \square

Proof of Theorem 3.1: Using Lemmas 3.1, 3.2, and the consequence of the Hilbert-Schmidt theorem for unbounded operators, it can be deduced that A_0 generates an infinite set of eigenvalues which are positive and increasing and a set of orthonormal eigenfunctions forming a basis for H . \square

The definition of the operator A_0 can help us to express the PDE in (5) and corresponding BCs in (6) (assuming zero inputs) in the form of an evolutionary equation in energy state space $\mathbb{H} = H_E^3(0,1) \times L^2(0,1)$; that is

$$\frac{d\xi(t)}{dt} = A\xi(t), \quad \xi(0) = \xi_0 \in D(A), \quad (11)$$

where $\xi(t) = (w, w_t)$, and the operator $A : D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is defined as follows:

$$\begin{aligned} A(f, g) &= (g, -A_0 f), \\ D(A) &= \{(f, g) \mid f \in D(A_0), g \in H_E^3(0,1)\}. \end{aligned} \quad (12)$$

The state space \mathbb{H} is equipped with an inner product induced norm defined as

$$\|(f, g)\|_{\mathbb{H}}^2 = \frac{1}{2} \int_0^1 \left\{ \left(f^{(2)} \right)^2 + \zeta \left(f^{(3)} \right)^2 + g^2 \right\} dx. \quad (13)$$

The operator A is skew-adjoint (i.e. $A^* = -A$) as a result of the symmetry and surjectivity of the operator A_0 . It can be readily found that the operator A has a compact resolvent due to the compactness of A_0^{-1} and the definition of $D(A)$ (see [40, Proposition 1]). The following proposition gives the relation between eigenvalues and eigenfunctions of the operators A_0 and A .

Proposition 3.1. Consider λ_i^2 and ϕ_i to be the eigenvalues and corresponding eigenfunctions of the operator A_0 , respectively. Then, the eigenvalues μ_k and the corresponding eigenfunctions ψ_k of A for $k \in \mathbb{Z}^* (= \mathbb{Z} \setminus \{0\})$ can be obtained from,

$$\begin{cases} \mu_k = i\lambda_k, & \lambda_{-k} = -\lambda_k, \\ \psi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\lambda_k} \phi_k \\ \phi_k \end{bmatrix}, & \phi_{-k} = -\phi_k. \end{cases} \quad (14)$$

Proof: Regardless of the definition of the operator A_0 , a proof of this proposition can be found in [38, Proposition 3.7.7]. \square

Now, we are able to determine the spectrum of the operator A_0 as presented in the following lemma.

Lemma 3.3. The spectrum $\sigma(A_0)$ of A_0 consists of isolated eigenvalues λ_n^2 which are geometrically simple and for sufficiently large positive integer n , admit the following asymptotic expression:

$$\left(\frac{27}{\zeta}\lambda_n^2 - \frac{2}{\zeta^3}\right)^{\frac{1}{6}} = \frac{3\pi}{\sqrt{3}}\left(n + \frac{1}{2}\right) + \frac{2}{\pi^2}\left(n + \frac{1}{2}\right)^{-2} + O(n^{-3}).$$

Proof: $\lambda^2 \in \mathbb{R}^+$ is an eigenvalue of A_0 iff there exist a $\phi \in D(A_0)$, $\phi \neq 0$ that satisfies

$$\begin{cases} \phi^{(4)} - \zeta\phi^{(6)} = \lambda^2\phi, \\ \phi(0) = \phi^{(1)}(0) = \phi^{(2)}(0) = 0, \\ \phi^{(3)}(1) = 0, \phi^{(2)}(1) - \zeta\phi^{(4)}(1) = 0, \\ \zeta\phi^{(5)}(1) - \phi^{(3)}(1) = 0. \end{cases} \quad (15)$$

In order to solve this boundary value problem, roots of the following characteristic polynomial has to be determined:

$$\zeta s^6 - s^4 + \lambda^2 = 0. \quad (16)$$

which yields six roots $\pm s_i$, $i = 1, 2, 3$. Subsequently, a fundamental solution to the ODE (15) is

$$\phi(x) = c_1 e^{s_1 x} + c_2 e^{-s_1 x} + c_3 e^{s_2 x} + c_4 e^{-s_2 x} + c_5 e^{s_3 x} + c_6 e^{-s_3 x}, \quad (17)$$

where constants c_i , $i = 1, 2, \dots, 6$, can be obtained by applying the BCs in (15) to $\phi(x)$; doing so leads to the following system of algebraic equations:

$$\mathbf{B}(s_i)[c_1, c_2, c_3, c_4, c_5, c_6]^T = \mathbf{0}_{6 \times 1}, \quad (18)$$

The determinant of $\mathbf{B}(s_i)$ must be zero so as to have a non-trivial solution for c_i . As a result, the eigenvalues can be extracted from the characteristic equation: $\det(\mathbf{B}(s_i)) = 0$. An asymptotic expression of the characteristic equation can be derived for sufficiently large λ . To this end, the solution s_i

to (16) as $\lambda \rightarrow \infty$ are approximated by

$$\begin{aligned} s_1^2 &= -\frac{1}{3\zeta}(27\zeta^2\lambda^2 - 2)^{\frac{1}{3}}, \\ s_2^2 &= \frac{1}{6\zeta}(27\zeta^2\lambda^2 - 2)^{\frac{1}{3}} + \frac{\sqrt{3}i}{6\zeta}(27\zeta^2\lambda^2 - 2)^{\frac{1}{3}}, \\ s_3^2 &= \frac{1}{6\zeta}(27\zeta^2\lambda^2 - 2)^{\frac{1}{3}} - \frac{\sqrt{3}i}{6\zeta}(27\zeta^2\lambda^2 - 2)^{\frac{1}{3}}. \end{aligned} \quad (19)$$

By substituting (19) into the characteristic equation and after performing some algebraic manipulations, the asymptotic expression of the characteristic equation for $\zeta = 1$ is derived as

$$\begin{aligned} F(\lambda^2) &= -\frac{1}{6}a^2(e^{2\bar{q}} + e^{-2\bar{q}} + e^{2q} + e^{-2q} + 8e^{\bar{q}} + 8e^{-\bar{q}} + 8e^q \\ &+ 8e^{-q} + 8e^{\bar{q}-q} + 8e^{q-\bar{q}} + e^{2\bar{q}-2q} + e^{2q-2\bar{q}} + 18) \\ &- m(e^{2\bar{q}} + e^{-2\bar{q}}) - \bar{m}(e^{-2q} + e^{2q}) - 2m(e^{\bar{q}} + e^{-\bar{q}}) \\ &- 2\bar{m}(e^{-q} + e^q) + 2(e^{\bar{q}-q} + e^{q-\bar{q}}) + e^{2\bar{q}-2q} + e^{2q-2\bar{q}} = 0, \end{aligned} \quad (20)$$

where $a = (27\lambda^2/\zeta - 2/\zeta^3)^{1/6}$, $q = (3/6 + \sqrt{3}i/6)a$, and $m = 1/2 + \sqrt{3}i/2$. In order to find an approximate solution to (20), all the terms in this equation need to be ordered according to their growth rate; that is

$$\begin{aligned} |a^2(e^{2\bar{q}} + e^{2q} + 8e^{\bar{q}} + 8e^q)| &\in \Theta(a^2 e^a), \\ |me^{2\bar{q}} + \bar{m}e^{2q} + 2me^{\bar{q}} + 2\bar{m}e^q| &\in \Theta(e^a), \\ &\vdots \end{aligned} \quad (21)$$

According to Rouché's theorem, the zeroes of the greatest term (i.e. those of order $a^2 e^a$) give the exact number of zeroes and also an estimation of zeroes of (20). Thus, we set

$$a^2(e^{2\bar{q}} + e^{2q} + 8e^{\bar{q}} + 8e^q) = 0,$$

which is simplified to

$$\left(\frac{\sqrt{3}}{3}a\right) + 8e^{-a/2} \cos\left(\frac{\sqrt{3}}{6}a\right) = 0. \quad (22)$$

Equation (22) admits the asymptotic solution $a_n = 3\pi(n + 1/2)/\sqrt{3} + \alpha_n$. In order to estimate α_n , the lower order terms (i.e. those of order e^a) need to be considered. In other words, by substituting a_n in

$$\begin{aligned} -\frac{1}{6}a^2(e^{2\bar{q}} + e^{2q} + 8e^{\bar{q}} + 8e^q) \\ - me^{2\bar{q}} - \bar{m}e^{2q} - 2me^{\bar{q}} - 2\bar{m}e^q = 0, \end{aligned}$$

which is simplified to

$$\begin{aligned} -\frac{1}{6}\left(\pi\left(n + \frac{1}{2}\right) + \frac{\sqrt{3}}{3}\alpha_n\right)^2 \sin\left(\frac{\sqrt{3}}{3}\alpha_n\right) \\ - \sin\left(\frac{\sqrt{3}}{3}\alpha_n\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}}{3}\alpha_n\right) = 0, \end{aligned}$$

and by using Maclaurin series, the above equation yields the solution $\alpha_n = 2(n\pi + \pi/2)^{-2} + \beta_n$, $\beta_n \in O(n^{-3})$. Therefore, we obtain

$$a_n = \frac{3\pi}{\sqrt{3}}\left(n + \frac{1}{2}\right) + \frac{2}{\pi^2}\left(n + \frac{1}{2}\right)^{-2} + \beta_n, \quad n \rightarrow \infty. \quad (23)$$

For the purposes of this paper, a better approximation is not needed.

In order to prove that the eigenvalues are geometrically simple, we first show by contradiction that $\phi^{(3)}(0)$ cannot be zero for an eigenfunction of the operator A_0 . To this end, multiply the ODE in (15) by $(x-1)\phi^{(1)}(x)$, integrate over x , and perform repeated integration by parts; it then follows that

$$\begin{aligned} & \int_0^1 (x-1)\phi^{(1)}(x) \left[\phi^{(4)}(x) - \zeta\phi^{(6)}(x) - \lambda^2\phi(x) \right] dx \\ &= \left[-\phi^{(1)}(x)\phi^{(2)}(x) + (x-1)\phi^{(1)}(x)\phi^{(3)}(x) \right. \\ &\quad \left. - \zeta(x-1)\phi^{(1)}(x)\phi^{(5)}(x) + \zeta\phi^{(1)}(x)\phi^{(4)}(x) \right. \\ &\quad \left. + 2\zeta(x-1)\phi^{(2)}(x)\phi^{(4)}(x) - 2\zeta\phi^{(2)}(x)\phi^{(3)}(x) \right]_0^1 \\ &\quad - \frac{1}{2} \left[(x-1)(\phi^{(2)}(x))^2 + \zeta(x-1)(\phi^{(3)}(x))^2 \right. \\ &\quad \left. + \lambda^2(x-1)(\phi(x))^2 \right]_0^1 + \frac{3}{2} \int_0^1 (\phi^{(2)}(x))^2 \\ &\quad + \frac{5\zeta}{2} \int_0^1 (\phi^{(3)}(x))^2 + \frac{\lambda^2}{2} \int_0^1 (\phi(x))^2. \end{aligned} \quad (24)$$

Now apply the BCs in (15) and assume $\phi^{(3)}(0) = 0$, we obtain

$$\frac{3}{2} \int_0^1 (\phi^{(2)}(x))^2 + \frac{5\zeta}{2} \int_0^1 (\phi^{(3)}(x))^2 + \frac{\lambda^2}{2} \int_0^1 (\phi(x))^2 = 0.$$

The above equation admits a unique solution $\phi(x) = 0$, which cannot be an eigenfunction of the invertible operator A_0 —this argument is suggested as an open problem in Guzmán and Zhu's paper [13]. Now, let ϕ_1 and ϕ_2 be two eigenfunctions of A_0 associated with the same eigenvalue λ^2 . Then, the function $\phi(x) = \phi_1^{(3)}(0)\phi_2(x) - \phi_2^{(3)}(0)\phi_1(x)$ satisfies (15) along with $\phi^{(3)}(0) = 0$. As shown, it follows that $\phi(x) = 0$, and thus the eigenfunctions ϕ_1 and ϕ_2 are not linearly independent. Therefore, the eigenvalues are all geometrically simple, and the proof is complete. \square

IV. ADMISSIBLE OBSERVATION OPERATOR AND EXACTLY OBSERVABLE SYSTEM

This section identifies those observation operators $C : D(A)(\subset \mathbb{H}) \rightarrow Y$ that are admissible and that define an exactly observable system. Roughly speaking, a system is said to be observable if all states can be determined through some partial measurements of states over a sufficiently long time interval; in addition, the concept of admissibility appears mainly in infinite dimensional dynamical system theory and shows that there exists an output function in $L^2([0, \infty), Y)$ for any initial state in \mathbb{H} . More precisely, the following definition has been introduced.

Definition 4.1. The operator $C : D(A)(\subset \mathbb{H}) \rightarrow Y$ is an admissible observation operator for the semigroup generated by A if there exist two positive constants M_1 and τ such that

$$\int_0^\tau \|CT(t)\xi_0\|_Y^2 dt \leq M_1 \|\xi_0\|_{\mathbb{H}}^2. \quad (25)$$

In addition, the pair (A, C) is exactly observable in time $t \geq \tau$ if there exists positive constant M_2 such that

$$\int_0^\tau \|CT(t)\xi_0\|_Y^2 dt \geq M_2 \|\xi_0\|_{\mathbb{H}}^2. \quad (26)$$

Various observation operators can be defined for the system; however, an in-domain point observation may not result in an exactly observable system. Focusing on non-located boundary observations, the physical properties that can be measured at the root of the beam are force, moment and non-classical moment. Hence, similar to the governing BCs defined in (6), the observation operators can be defined as follows:

$$\begin{cases} C_1\xi(t) := F_0 = \zeta \frac{\partial^5 w}{\partial x^5}(0, t) - \frac{\partial^3 w}{\partial x^3}(0, t), \\ C_2\xi(t) := M_0^c = \frac{\partial^2 w}{\partial x^2}(0, t) - \zeta \frac{\partial^4 w}{\partial x^4}(0, t), \\ C_3\xi(t) := M_0^{nc} = \zeta \frac{\partial^3 w}{\partial x^3}(0, t). \end{cases} \quad (27)$$

A. Time domain approach

In the existing literature, admissibility of an observation operator and exact observability of a system are commonly tested by resorting to the basic definition of admissibility and exact observability. In this way, the governing equation is manipulated by performing some integration by parts and using some well-known inequalities in order to construct the desired inequalities in Definition 4.1 (see e.g. [19], the multiplier method). In what follows, this method is used to prove the admissibility of the observation operator C_3 and the exact observability of pair (A, C_3) .

Theorem 4.1. The operator C_3 is an admissible observation operator; moreover, the pair (A, C_3) is exactly observable.

Proof: To prove the theorem, we first multiply the governing equation in (5) by the term $(1-x)\frac{\partial w}{\partial x}$ and integrate with respect to x and t ; that is

$$\int_0^T \int_0^1 (1-x)w_1(w_4 - \zeta w_6 + \ddot{w})dx dt = 0, \quad (28)$$

where w_m denotes the m th order derivative with respect to x , and \dot{w} stands for the derivative of w with respect to time. By performing repeated integration by parts for each integral terms of the above equation and eliminating double integral terms as far as possible, the following equality is derived:

$$\begin{aligned} & \int_0^T [(1-x)w_1(w_3 - \zeta w_5)]_0^1 dt + \int_0^T [w_1(w_2 - \zeta w_4)]_0^1 dt \\ & - \int_0^T \int_0^1 \left[\frac{1}{2}\dot{w}^2 + \frac{3}{2}(w_2)^2 + \frac{5\zeta}{2}(w_3)^2 \right] dx dt \\ & + \frac{1}{2} \int_0^T [(1-x)w\ddot{w}]_0^1 dt - \frac{1}{2} \left[[(1-x)\dot{w}_1w]_0^1 \right]_0^T \\ & + \int_0^1 [(1-x)w_1\dot{w}]_0^T dx - \frac{1}{2} \int_0^T [\zeta(x-1)w_2w_4]_0^1 dt \\ & + \frac{1}{2} \int_0^T [\zeta w_2w_3]_0^1 dt + \frac{1}{2} \int_0^T [\zeta(x-1)(w_3)^2]_0^1 dt \\ & - \frac{1}{2} \int_0^T [2\zeta w_3w_2]_0^1 dt - \frac{1}{2} \int_0^T [\zeta(x-1)w_4w_2]_0^1 dt \\ & + \int_0^T [\zeta w_3w_2]_0^1 dt + \frac{3\zeta}{2} \int_0^T [w_2w_3]_0^1 dt \\ & + \frac{1}{2} \int_0^T [(x-1)(w_2)^2]_0^1 dt = 0. \end{aligned}$$

Applying the BCs in (6) (inputs are set to zero) to the previous equality reduces this equation to

$$\int_0^T \zeta(w_3)^2 \Big|_{x=0} dt = \int_0^T \int_0^1 \left[\dot{w}^2 + 3(w_2)^2 + 5\zeta(w_3)^2 \right] dx dt + 2 \int_0^1 [(x-1)w_1\dot{w}]_0^T dx. \quad (29)$$

The next step is to find an upper and lower bound for the right hand side expressions. Focusing on the third integral, one can apply triangular inequality and then Young's inequality to obtain the following inequality:

$$\left| 2 \int_0^1 [(x-1)w_1\dot{w}]_0^T dx \right| \leq \int_0^1 |x-1| \left[(w_1)^2 + \dot{w}^2 \right]_{t=0} dx + \int_0^1 |x-1| \left[(w_1)^2 + \dot{w}^2 \right]_{t=T} dx.$$

Since the maximum value of $|x-1|$ in the interval $[0, 1]$ is one, this term can be dropped from the above inequality. Afterwards, a one dimensional version of Poincaré inequality [41, Lemma 2.1] can be used to get

$$\left| 2 \int_0^1 [(x-1)w_1\dot{w}]_0^T dx \right| \leq \int_0^1 \left[4(w_2)^2 + \dot{w}^2 \right]_{t=0} dx + \int_0^1 \left[4(w_2)^2 + \dot{w}^2 \right]_{t=T} dx. \quad (30)$$

The integrals including w_2^2 are bounded by the value of the strain energy of the system; this bound can be found by applying Poincaré inequality to the expression of the strain energy in (7). That is

$$\int_0^1 (w_2)^2 dx \leq 2(w_2)^2 \Big|_{x=0} + \int_0^1 4(w_3)^2 dx \Rightarrow \frac{1}{2} \int_0^1 \left[\left(1 + \frac{\zeta}{4}\right) (w_2)^2 \right] dx \leq U. \quad (31)$$

Consequently, by substituting (31) into (30) and using the definition of the kinetic energy in (7), it follows that

$$\left| 2 \int_0^1 [(x-1)w_1\dot{w}]_0^T dx \right| \leq \left[\frac{32U}{4+\zeta} + 2K \right]_{t=0} + \left[\frac{32U}{4+\zeta} + 2K \right]_{t=T} \leq 2 \max\left(2, \frac{32}{4+\zeta}\right) E. \quad (32)$$

Returning to (29), for the second integral on the right hand side of this equation, an upper and lower bounds can be readily found as follows:

$$2ET \leq \int_0^T \int_0^1 \left(\dot{w}^2 + 3(w_2)^2 + 5\zeta(w_3)^2 \right) dx dt \leq 10ET. \quad (33)$$

Finally, combining (29), (32), and (33), we obtain:

$$2 \left(T - \max\left(2, \frac{32}{4+\zeta}\right) \right) E \leq \int_0^T \zeta(w_3)^2 \Big|_{x=0} dt \leq \left(10T + 2 \max\left(2, \frac{32}{4+\zeta}\right) \right) E.$$

According to the definitions of the observation operator C_3 , the total energy, the state space, and the corresponding inner product induced norm, the previous inequality can be rewritten as:

$$2\zeta \left(T - \max\left(2, \frac{32}{4+\zeta}\right) \right) \|\xi(0)\|_{\mathbb{H}}^2 \leq \int_0^T |C_3\xi(t)|^2 dt \leq \zeta \left(10T + 2 \max\left(2, \frac{32}{4+\zeta}\right) \right) \|\xi(0)\|_{\mathbb{H}}^2,$$

Considering Definition 4.1, it is sufficient to choose $\tau = T > \max(2, 32/(4+\zeta))$ to complete the proof. \square

As can be seen, this usual way of proving an observability estimate is rather constructive and cannot easily be used to show that a system with an observation operator is not exactly observable. In addition, unlike the frequency domain approach, the time domain approach will not yield an optimal observability time τ . Accordingly, in the following, the frequency domain approach is adopted to show that the observation operator C_1 and C_2 are not admissible and that the optimal observability time τ for the observation operator C_3 is in fact zero.

B. Frequency domain approach

Another way to tackle the observability problem for a given observation operator is to consider the image of eigenfunctions of the state operator under the observation operator, providing that the state operator is diagonalizable, which is the case in most physical systems [38]. The following proposition provides a powerful tool for studying the observability problem of such systems.

Proposition 4.1. Assume that the operator A is skew-adjoint and has a compact resolvent, denoting by ψ_k the eigenfunctions and by $i\lambda_k$ the eigenvalues of A that are simple and ordered such that the sequence λ_k is strictly increasing. Then, the operator $C : D(A)(\subset \mathbb{H}) \rightarrow Y$ is an admissible observation operator for the semigroup generated by A , and the pair (A, C) is exactly observable in any time $\tau > 0$ if $\lim_{|k| \rightarrow \infty} (\lambda_{k+1} - \lambda_k) = \infty$ and there exist two positive constants β_1 and β_2 such that

$$\beta_1 \leq \|C\psi_k\|_Y \leq \beta_2, \quad \forall k \in \mathbb{Z}^*. \quad (34)$$

Proof: A proof has been presented in [38, Corollary 6.9.6] by utilizing a wave packets concept. \square

It is worth mentioning that the inequality $\|C\psi_k\|_Y \leq \beta_2$ solely guarantees the admissibility of an observation operator.

Remark 4.1. For a finite dimensional dynamical system, every observation operator is bounded and hence admissible.

To derive an estimate similar to (34) for our system, we need to determine the asymptotic behavior of the coefficients $c_{i,n}$, $i = 1, 2, \dots, 6$, in (17) corresponding to the eigenfunction $\phi_n(x)$. These asymptotic expressions can be computed with the aid of a symbolic computation package such as MATLAB Symbolic Toolbox. It is observed that

$$\begin{aligned} \|\phi_n(x)\|_2 &\in \Theta(a_n^{12} e^{a_n}), \quad \phi_n^{(3)}(0) \in \Theta(a_n^{15} e^{a_n}), \\ \phi_n^{(4)}(0) &\in \Theta(a_n^{16} e^{a_n}), \quad \phi_n^{(5)}(0) \in \Theta(a_n^{17} e^{a_n}), \end{aligned}$$

where a_n is given in (23). Subsequently, images of the eigenfunctions of A under the observation operators (27) can be obtained by using Proposition 3.1. Then, it is seen that

$$\lim_{|k| \rightarrow \infty} |C_1 \psi_k| = +\infty, \quad \lim_{|k| \rightarrow \infty} |C_2 \psi_k| = +\infty, \\ \lim_{|k| \rightarrow \infty} |C_3 \psi_k| = \sqrt{3\zeta}.$$

These results suggest that the observation operators C_1 and C_2 are not admissible since the images of the eigenfunctions of A are not bounded under these operators. On the other hand, the observation operator C_3 is an admissible observation operator and defines an exactly observable system; this statement is proved in the following theorem.

Theorem 4.2. The observation operator C_3 is admissible for the semigroup generated by A ; moreover, the pair (A, C_3) is exactly observable in any time $\tau > 0$.

Proof: In the previous section, it is shown that the operator A is skew-adjoint and has a compact resolvent. Furthermore, according to Lemma 3.3, for sufficiently large k , the sequence λ_k is of order exactly k^3 , and thus $\lim_{|k| \rightarrow \infty} (\lambda_{k+1} - \lambda_k) = +\infty$ holds regardless of the choice of an observation operator—this property of the eigenvalues of the micro-beam state operator is suggested as an open problem in Guzmán and Zhu's paper [13].

Now, it is sufficient to show that the sequence $|C_3 \psi_k|$ is bounded below and above by some positive numbers. To this end, let ϕ_k 's be normalized eigenfunctions of A_0 , apply the BCs in (15) to (24), we obtain

$$\zeta \left(\phi_k^{(3)}(0) \right)^2 = \lambda_k^2 + \int_0^1 5\zeta \left(\phi_k^{(3)}(x) \right)^2 + 3 \left(\phi_k^{(2)}(x) \right)^2 dx. \quad (35)$$

The following integration by parts will then help us to find an estimate for the above integral term:

$$\begin{aligned} & \int_0^1 \phi_k(x) \left[\phi_k^{(4)}(x) - \zeta \phi_k^{(6)}(x) - \lambda_k^2 \phi_k(x) \right] dx \\ &= -\lambda_k^2 + \zeta \int_0^1 \left(\phi_k^{(3)}(x) \right)^2 dx + \int_0^1 \left(\phi_k^{(2)}(x) \right)^2 dx \\ &+ \left[\phi_k(x) \left(\phi_k^{(3)}(x) - \zeta \phi_k^{(5)}(x) \right) - \zeta \phi_k^{(2)}(x) \phi_k^{(3)}(x) \right. \\ &\quad \left. + \phi_k^{(1)}(x) \left(\zeta \phi_k^{(4)}(x) - \phi_k^{(2)}(x) \right) \right]_0^1 = 0. \end{aligned}$$

Applying the BCs in (15) to the above equation yields

$$\int_0^1 \zeta \left(\phi_k^{(3)}(x) \right)^2 + \left(\phi_k^{(2)}(x) \right)^2 dx = \lambda_k^2. \quad (36)$$

Combining (35) and (36), we obtain

$$4\lambda_k^2 \leq \zeta \left(\phi_k^{(3)}(0) \right)^2 \leq 6\lambda_k^2. \quad (37)$$

Consequently, applying Proposition 3.1, it follows that $\sqrt{2\zeta} \leq |C_3 \psi_k| \leq \sqrt{3\zeta}$. Therefore, according to Proposition 4.1, the observation operator C_3 is admissible, and the pair (A, C_3) is exactly observable in any time $\tau > 0$. \square

Knowing that the system is exactly controllable and observable, future research will be focused on designing an exponentially stable observer-base controller. Afterwards, the infinite dimensional observer has to be truncated for practical

applications. This late-lumping approach, where an infinite dimensional observer is designed to be reduced to a finite dimensional observer, has several advantages over the early-lumping approach. However, it poses some problems that must be addressed in future work.

V. CONCLUSION

The exact observability of a flexible strain gradient micro-beam was studied with an investigation of different observation operators. It was shown that only the measurement of the non-classical moment at the root of the beam yields an admissible observation operator and defines an exactly observable system. This work contributes to the existing literature by considering a more realistic mathematical model for micro-scale flexible beams as well as adopting and comparing two different approaches to tackling the observability problem.

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